Contents

Introduction i

Foreword i
Scientific Context i
Goals and Organization of the Material iv
Notation and common Expressions v
Acknowledgments vi

1 Preliminaries 1

Systems 1
Proof Interpretation 2
1.1 The System WE-HA$^\omega$ 3
1.2 Fragments and Extensions of WE-HA$^\omega$ 7
1.3 The ordered Field of rational Numbers within WE-HA$^\omega$ 8
1.4 The Archimedean ordered Field of real Numbers within WE-HA$^\omega$ . . . 8
1.5 Kuroda's negative Translation 10
1.6 Gödel's Functional Interpretation 10
1.7 Majorizability 14
1.8 Monotone Functional Interpretation 14
1.9 A specific metatheorem 16
1.10 Models of E-PA$^\omega$ 17
1.11 Bar Recursion 18
1.12 Law of excluded Middle 20
1.13 Double Negation Shift 21
1.14 Axiom of Choice 22
1.15 Arithmetical Comprehension 24
1.16 Weak König's Lemma 30
1.17 Complexity classes and bar recursion 41

2 Effective learnability 45

2.1 Fluctuations versus effective learnability 47
2.2 Cauchy statements and unrestricted use of Σ_1^0-LEM 60
2.3 Which Cauchy statements are effectively learnable and which are not 63
2.4 When learnability implies fluctuation bounds 74

3 Interpreting a strong non-linear ergodic theorem **79**

3.1 Arithmetizing Wittmann's proof 82
3.2 Obtaining a bound 85
3.3 A general bound existence theorem 86
3.4 Uniform bounds for Wittmann's ergodic theorems 89
3.5 Proof of Proposition 3.9 92

4 Interpreting Bolzano-Weierstraß **99**

4.1 Interpreting Bolzano-Weierstraß 101
4.2 Analysis of the Complexity of the Realizers 114

5 Functional interpretation for IST **119**

5.1 Formalities 120
5.2 Nonstandard principles 124
5.3 A functional interpretation for E-$\mathrm{HA}^{\omega *}_{\mathrm{st}}$ 128
5.4 The system E-$\mathrm{PA}^{\omega *}_{\mathrm{st}}$ and negative translation 131
5.5 A functional interpretation for E-$\mathrm{PA}^{\omega *}_{\mathrm{st}}$ 132
5.6 Countable saturation 137

Zusammenfassung **141**

References **143**