Preface

This work offers a comprehensive, clearly written presentation of dynamical systems and pertinent mathematics. As only one of its novel features, it contains two-sided bounds on the solution of stability problems leading to important new results and to significant improvements compared to results obtained by the Lyapunov method.

All started with a closer look to the last one which can be described as follows.

The Lyapunov method for the determination of the stability behavior of a dynamical system consists essentially in two steps, namely first, in establishing an energy function (also called Lyapunov function) $V(t) = E(t)$ pertinent to the studied dynamical system and, second, in drawing conclusions on the stability behavior from the time derivative $\dot{V}(t)$. Roughly speaking, $\dot{V}(t) < 0$ implies asymptotic stability, $\dot{V}(t) \leq 0$ stability, and $\dot{V}(t) > 0$ instability. In this way, many stability problems can be solved.

However, some shortcomings of the Lyapunov method are known. For example, in a linear one-mass vibration model with moderate damping, for the energy function $V(t)$, one obtains only $\dot{V}(t) \leq 0$, i.e., only stability of the system even though one can easily find the exact solution showing that the pertinent dynamical system is asymptotically stable.

This shortcoming is described, for instance, in Hagedorn [21, Section 2.3, pp. 87-88]. In order to remedy this problem, the author used

- the state-space description of the associated differential equations,
- the special weighted norm $\|\cdot\|_R$ derived by the author in previous work, and
- the equivalence of norms in finite-dimensional spaces.

It turned out that, by these Key Mathematical Tools, also similar higher-dimensional dynamical systems can be successfully treated.

Further, in addition to the asymptotic stability of such systems, two-sided bounds on the state-space solution vector can be obtained.

Another shortcoming of the Lyapunov method is that, for some nonlinear systems, particular integrals of the associated system of differential equations are needed. But, it is often difficult or impossible to find these particular solutions.

We will show in this research-oriented monograph that one can overcome also the last-mentioned disadvantage of the Lyapunov method and demonstrate this by various examples. *More precisely, we apply the new theory of stability developed in this book to a series of examples that other authors treat by the Lyapunov method and compare the results which show that our theory of stability is superior to the Lyapunov method.*

Dynamical systems in this monograph are described by ordinary differential equations in the vector form $\dot{x}(t) = f(t, x(t))$ and associated initial conditions $x(t_0) = x_0 \neq 0$ in the space of n-tuples denoted by $\mathbb{F}^n$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, in other words, by initial value problems. The right-hand member $f(t, u) \in \mathbb{F}^n$ will be defined for the infinite range of values t, namely for $t \geq t_0$ and for $u \in G_0$ where G_0 is a convex closed subset of $\mathbb{F}^n$ containing the zero vector 0. As opposed to this, the existence and uniqueness of the solution vector $x(t)$ will be proven for a finite range of values t given by $t_0 \leq t \leq t_1$.

Under additional conditions, it will be possible to choose t_1 arbitrarily large in which case the solution vector $x(t)$ exists even for the infinite range $t \geq t_0$.

As prerequisites for the understanding, this monograph requires knowledge imparted in courses such as Calculus, Linear Algebra, and Ordinary Differential Equations.

Basic acquaintance in Functional Analysis is advantageous, but not necessary. By using functional-analytic methods, it is possible to obtain a unified treatment of the Theory of Dynamical Systems which is one of the main objectives.

This book is not only a monograph, but also a textbook. Consequently, it should be appropriate for teaching. Therefore, the style of the book is expository. So, also readers without knowledge of functional-analytic methods should be able to profit by this book.
As a rule, the proofs and derivations are at great length. But, some results are only referred to. Examples for this are a theorem on the eigenvalues and eigenvectors of symmetric mappings, a representation formula for the fundamental matrix of a periodic matrix, and the Theorem on Continuous Dependence on the solution of a nonlinear ordinary differential equation. Further, the eigenvalues and eigenvectors of some matrices are merely stated in which case their determination is left to the reader. Moreover, in rare cases, some lengthy derivations are also left to the reader which should pose no problem, however. On the whole, the book is as far as possible self-contained.

Further, it is intended to be an introduction to the subject for Mathematicians, on the one hand, as well as for Physicists and Engineers, on the other hand. In order to take this into account, for Mathematicians, we have included hints for the derivation or the derivations themselves of the differential equations that describe the occurring dynamical systems, and for Physicists and Engineers, we have given detailed mathematical derivations and explanations.

The Important Classes of Dynamical Systems that are treated in this book are

- Linear Autonomous Systems,
- Linear Periodic Systems, and
- Nonlinear Systems such as Quasi-Linear Systems with Autonomous or Periodic Linear Part.

For Linear Autonomous Systems, the right-hand member $f(t,x)$ has the form $f(t,x) = A\,x$ where A is a constant matrix.

For Linear Periodic Systems, $f(t,x)$ is given by $f(t,x) = A(t)\,x$ where $A(t)$ is a periodic matrix.

In the case of Nonlinear Systems, we study

- systems with autonomous linear part where $f(t,x)$ has the form $f(t,x) = A\,x + h(x)$ with nonlinear vector function $h(x)$,
- systems with periodic linear part where $f(t,x)$ has the form $f(t,x) = A(t)\,x + h(x)$ and where $A(t)$ is a periodic matrix, and
- special nonlinear systems whose right-hand members have special form.

We mention that $h(x)$ may be replaced by $h(t,x)$ if the conditions on h are adapted to this change.

Further, each of the above sections is complemented by a section entitled Problems and Solutions at the end of each chapter.

The studied dynamical systems can be described as special cases of the general initial value problem $\dot{x}(t) = f(t,x(t)),\ x(t_0) = x_0 \neq 0$.

The existence and uniqueness of its solution is proven by the Contraction Mapping Theorem in an appropriate Banach space. Since this Banach space is infinite-dimensional, not all norms in it are equivalent. But, it is possible to define equivalent norms which is made use of in the proof of the completeness of the considered space.

This book contains research results obtained by the author.

Some of them were published previously such as the derivation of the special weighted norm $\|\cdot\|_R$ and two-sided bounds on the solution vector $x(t)$.

However, many of them were not published before and contain significant improvements compared to the Lyapunov method.

New results published for the first time in this book show that, for instance,

- improved two-sided bounds on the solution vector $x(t)$ can be obtained
- the derivation of two-sided bounds on the solution vector $x_p(t)$ of periodic linear systems can be reduced to that of the autonomous case
- the derivation of two-sided bounds on the solution vector $x(t)$ of nonlinear systems with periodic linear part can be reduced to that with autonomous linear part, and
- novel techniques could be developed for problems of the form $\dot{x} = f(x)$ with $f(x) = A\,x + h(x)$ where $A = f'(0)$ with $\nu[A] = \nu_{x_0}[A] = \nu[f'(0)] = 0$, where $\nu[A]$ is the spectral abscissa of matrix A and $\nu_{x_0}[A]$ is the spectral abscissa of A with respect to the vector $x_0 \neq 0$, and where $f'(0)$ is the Jacobi matrix of f at $t = 0$

In other words, in addition to the advantages of the new theory of stability mentioned already over the Lyapunov method, the above systems are analyzed in the same way which was also not done before.

We mention that the developed theory cannot only be applied to Linear and Nonlinear Dynamical Systems, but also to Linear and Nonlinear Control Problems.

Denotations or expressions used in this Preface not familiar to the reader will be explained, in due course.

As is custom, a differential equation $\dot{x}(t) = f(t, x(t))$ is often written in the form $\dot{x} = f(t, x)$. In addition, we sometimes write $\dot{x}(t) = f(t, x)$ or the like. Often we speak of the function $x(t)$ even though we have written down the function value. In this case, we mean the function x and that it depends on the time variable t.

In the case of other inaccuracies, it will become clear from the context what is meant.

If a dynamical system is *stable, but not asymptotically stable*, we abbreviate this by *stable (b.n.as.st.)*.

If not specified, $\|\cdot\|$ stands for an arbitrary norm and $\mathbb{F}$ for the field of real or complex numbers, i.e., for $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, as the case may be.

The transpose of an $m \times n$ matrix $B = (b_{jk})$ is denoted by B^T and is the $n \times m$ matrix defined by $B^T = (b^T_{jk})$ with $b^T_{jk} = b_{kj},\ j = 1, \ldots, m,\ k = 1, \ldots, n.$

There is also an alphabetic index.

More information on the subject of the book can be found in the detailed Table of Contents.

At this point, the author would like to thank the referees very much for their positive assessments of this book.

He also wants to give thanks to Logos Verlag for the pleasant collaboration.

The author would like to stress, however, that the responsibility for the content of this book lies solely with him.

Proposals for improvements of this book by readers are welcome.

Berlin, April 2025 Ludwig Kohaupt

Part I

Preliminaries

1 Key Mathematical Tools

Chapter 1 consists of the key mathematical tools used throughout this book and is subdivided into the following 5 Sections:

1.1 Equivalence of Norms in Finite-Dimensional Spaces

1.2 Special Weighted Norm $\|\cdot\|_R$

1.3 State-Space Description of Dynamical Systems

1.4 Symmetric Matrices and Mappings

1.5 Problems and Solutions

More details on their contents can be found at the beginning of each of the individual 5 sections.

Of particular importance for this book are the first three Sections 1.1, 1.2, and 1.3.

1.1 Equivalence of Norms in Finite-Dimensional Spaces

The 1st section of Chapter **1**, namely Section **1**.1, on the equivalence of norms in finite-dimensional spaces, contains the following subsections:

1.1.1 Vector norms

1.1.2 Matrix norms

1.1.3 Scalar products, weighted scalar products, and weighted norms

Subsection 1.1.1 assembles the known properties of vector norms. Most important is Theorem 1.1.1 on the equivalence of the usual norm $\|\cdot\|_2$ and every norm $\|\cdot\|$ on $\mathbb{F}^n$.

Subsection 1.1.2 collects the properties of matrix norms. The equivalence of matrix norms is established by regarding $m \times n$ matrices as column vectors on the space $\mathbb{F}^N$ with $N = m\,n$. This is similar to the column-wise storage of matrices in computer programs such as Matlab.

Subsection 1.1.3 describes the properties of scalar products, weighted scalar products, and weighted norms.
If $(\cdot,\cdot)$ is a scalar product for $\mathbb{F}^n$, then a weighted scalar product is given by $(u,v)_C = (C\,u, v)$, $u, v \in \mathbb{F}^n$ where $C \in \mathbb{F}^{n\times n}$ is a positive definite matrix.

The most important result of this subsection is Theorem 1.1.4 that states that every scalar product $(\cdot,\cdot)$ for $\mathbb{F}^n$ can be expressed as $(u,v) = (u,v)_C = (C\,u,v)_2$, $u,v \in \mathbb{F}^n$ for some positive definite matrix C.
As a consequence, for the weighted norm $\|\cdot\|_C$ induced by the weighted scalar product $(\cdot,\cdot)_C$, one obtains the equivalence of the norms $\|\cdot\|_2$ and $\|\cdot\|_C$ as well as the generalized Schwarz inequality $|(u,v)_C| \le \|u\|_C\,\|v\|_C$ $u,v \in \mathbb{F}^n$.

Now, the individual subsections follow.

1.1.1 Vector norms

The content of this subsection is taken almost verbatim from [74, Section 5.1].
Let E be a linear space or vector space over the field $\mathbb{F} = \mathbb{R}$ of real numbers or the field $\mathbb{F} = \mathbb{C}$ of complex numbers. Then, the vector space E is said to be *normed* or a *normed space* if there is associated with each vector $u \in E$ a real number $\|u\|$, called *norm* of u, with the following properties:

$$(N1) \qquad \|u\| \ge 0, \quad u \in E$$

$$(N2) \qquad \|u\| = 0 \Longleftrightarrow u = 0, \quad u \in E$$

$$(N3) \qquad \|\lambda u\| = |\lambda|\,\|u\|, \quad \lambda \in \mathbb{F}, \quad u \in E$$

$$(N4) \qquad \|u+v\| \le \|u\| + \|v\|, \quad u,\, v \in E$$

The inequality $(N4)$ is called the *triangle inequality*, and from it immediately follows the further inequality

$$|\,\|u\| - \|v\|\,| \le \|u-v\|, \quad u,v \in E. \tag{1.1.1}$$

A very simple example of a vector space is the n-dimensional space of n-tuples $\mathbb{F}^n = \mathbb{R}^n$ or $\mathbb{F}^n = \mathbb{C}^n$, where n is a natural number.
As is known, addition of vectors and multiplication of vectors by numbers are defined by

$$u+v = (u_1+v_1,\ldots,u_n+v_n), \qquad \lambda u = (\lambda u_1,\ldots,\lambda u_n)$$

for $u = (u_1,\ldots,u_n)$, $\quad v = (v_1,\ldots,v_n) \in \mathbb{F}^n$, $\lambda \in \mathbb{F}$.
Important norms on $\mathbb{F}^n$ are the *maximum norm* $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \max_{j=1,\ldots,n} |u_j|, \tag{1.1.2}$$

the *Euclidean* or *unitary norm* $\|\cdot\|_2$ defined by

$$\|u\|_2 = (\sum_{j=1}^{n} |u_j|^2)^{\frac{1}{2}}, \tag{1.1.3}$$

and the norm $\|\cdot\|_1$ defined by

$$\|u\|_1 = \sum_{j=1}^{n} |u_j| \tag{1.1.4}$$

for $u = (u_1, \ldots, u_n) \in \mathbb{F}^n$.
For these norms, the properties $(N1) - (N3)$ above follow immediately from the definitions, as does the triangle inequality $(N4)$ for the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$.
Finally, we show $(N4)$ for the Euclidean or unitary norm $\|\cdot\|_2$. The Euclidean or unitary norm is obtained for $\mathbb{F} = \mathbb{R}$ from the scalar product

$$(u, v)_2 = \sum_{j=1}^{n} u_j v_j, \quad u = (u_1, \ldots, u_n), \quad v = (v_1, \ldots, v_n) \in \mathbb{R}^n \tag{1.1.5}$$

or, for $\mathbb{F} = \mathbb{C}$, from the scalar product

$$(u, v)_2 = \sum_{j=1}^{n} u_j \bar{v}_j, \quad u = (u_1, \ldots, u_n), \quad v = (v_1, \ldots, v_n) \in \mathbb{C}^n \tag{1.1.6}$$

by writing

$$\|u\|_2 = \sqrt{(u, u)_2}\,, \tag{1.1.7}$$

the bar in (1.1.6) meaning the complex-conjugate.
The scalar product $(\cdot, \cdot)_2$ on $\mathbb{F}^n$ always satisfies the *Schwarz inequality*

$$|(u, v)_2| \leq \|u\|_2 \|v\|_2, \quad u, v \in \mathbb{F}^n \tag{1.1.8}$$

and so, in this case, the *triangle inequality* follows from the relation

$$\|u+v\|_2^2 = (u+v, u+v)_2 \leq \|u\|_2^2 + \|v\|_2^2 + 2\|u\|_2\|v\|_2 = (\|u\|_2 + \|v\|_2)^2, \quad u, v \in \mathbb{F}^n.$$

Familiar concepts from elementary classical geometry such as point, distance, spherical surface, etc., can also be applied to the vector space $\mathbb{F}^n$ with every norm. In this geometric language, we refer to vectors u, v in $\mathbb{F}^n$ as *points*, and $\|u - v\|$ as the *distance* between the points u and v. Further, for a point c in $\mathbb{F}^n$ and a positive number ϱ, the set of points

$$B_\varrho(c) = \{u \in \mathbb{F}^n \mid \|u - c\| \leq \varrho\} \tag{1.1.9}$$

is called the *closed ball* with center c and radius ϱ. Similarly, the open ball is defined by the set

$$\dot{B}_\varrho(c) = \{u \in \mathbb{F}^n \mid \|u - c\| < \varrho\} \tag{1.1.10}$$

and the *spherical surface* with center c and radius ϱ as the set

$$S_\varrho(c) = \{u \in \mathbb{F}^n \mid \|u - c\| = \varrho\}. \tag{1.1.11}$$

Example. The maximum norm $\|\cdot\|_\infty$ on the n-dimensional space $\mathbb{F}^n$ is of particular importance for applications. In this case, when $\mathbb{F}^n = \mathbb{R}^n$, and $c \in \mathbb{R}^n$ and $\varrho > 0$, the "ball" $B_\varrho(c)$ is simply the n-dimensional interval $B_\varrho(c) = [c-\varrho, c+\varrho]$ since for $u = (u_1, \ldots, u_n)$ in $B_\varrho(c)$ with $c = (c_1, \ldots, c_n)$, the relation

$$\max_{j=1,\ldots,n} |u_j - c_j| \leq \varrho \quad \text{or} \quad c_j - \varrho \leq u_j \leq c_j + \varrho, \quad j = 1, \ldots, n \tag{1.1.12}$$

holds. □

In the next theorem, we prove an important inequality for norms on $\mathbb{F}^n$.

Theorem 1.1.1. *(Equivalence of vector norms in finite-dimensional spaces) Let $\|\cdot\|$ be an arbitrary norm on the n-dimensional space $\mathbb{F}^n$, and let $\|\cdot\|_2$ be the Euclidean or unitary norm. Then, there exist positive constants γ_0, γ_1 such that*

$$\gamma_0 \|u\|_2 \leq \|u\| \leq \gamma_1 \|u\|_2, \quad u \in \mathbb{F}^n. \tag{1.1.13}$$

Proof. Let $e_1, \ldots, e_n$ be the basis vectors

$$e_j = (\delta_{1j}, \ldots, \delta_{nj}), \quad j = 1, \ldots, n \tag{1.1.14}$$

in the n-dimensional space $\mathbb{F}^n$ where $\delta_{ij},\ i, j = 1, \ldots, n$ is the Kronecker symbol. Then, every vector $u \in \mathbb{F}^n$ can be expressed as

$$u = (u_1, \ldots, u_n) = \sum_{j=1}^{n} u_j e_j. \tag{1.1.15}$$

So, the norm $\|\cdot\|$ satisfies the estimate

$$\|u\| \leq \sum_{j=1}^{n} |u_j| \, \|e_j\| \leq \gamma_1 \|u\|_2, \quad u \in \mathbb{F}^n, \tag{1.1.16}$$

with the constant

$$\gamma_1 = (\sum_{j=1}^{n} \|e_j\|^2)^{\frac{1}{2}}. \tag{1.1.17}$$

Moreover, every norm $\|\cdot\|$ is a Lipschitz-continuous function, and therefore in particular, a continuous real-valued function

$$h(u) = \|u\|, \quad u \in \mathbb{F}^n, \tag{1.1.18}$$

because of the inequality

$$|h(u) - h(v)| = |\,\|u\| - \|v\|\,| \leq \|u - v\| \leq \gamma_1 \|u - v\|_2, \quad u, v \in \mathbb{F}^n. \tag{1.1.19}$$

In the n-dimensional space $\mathbb{F}^n$, the unit sphere $S = S_1(0) = \{u \in \mathbb{F}^n \mid \|u\| = 1\}$ is bounded and closed. Therefore, by the *Weierstrass theorem*, the real-valued continuous function $h(u) = \|u\|$ on S has a minimum at some point $w \in S$ so that

$$\gamma_0 = h(w) = \min_{\|u\|_2=1} h(u) \leq h(v), \quad v \in S. \tag{1.1.20}$$

Here, γ_0 is necessarily positive since otherwise we should have

$$\gamma_0 = h(w) = \|w\| = 0, \tag{1.1.21}$$

and therefore also $w = 0$, as opposed to $w \in S$ or $\|w\|_2 = 1$. Finally, for every $u \neq 0$, we have $v = (1/\|u\|_2)\, u \in S$, and therefore,

$$\gamma_0 = h(w) \leq h(v) = h(u/\|u\|_2) = \frac{\|u\|}{\|u\|_2},$$

i.e.,

$$\gamma_0 \|u\|_2 \leq \|u\|, \quad u \in \mathbb{F}^n. \tag{1.1.22}$$

On the whole, Theorem 1.1.1 is proven. □

Remark. Instead of writing $u \in \mathbb{F}^n$ in the form (1.1.15), i.e., as a row vector with parentheses, we also write it in the form of a column vector. Sometimes, we use brackets instead of parentheses. So, the following writings may occur:

$$\begin{aligned} u &= (u_1, \ldots, u_n), \\ u &= [u_1, \ldots, u_n], \\ u &= \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \\ u &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \end{aligned}$$

to denote vectors $u \in \mathbb{F}^n$. □

By Theorem 1.1.1, every norm on the n-dimensional space $\mathbb{F}^n$ is equivalent to the Euclidean or unitary norm $\|\cdot\|_2$. For the maximum norm, we clearly have

$$\|u\|_\infty \leq \|u\|_2 \leq \sqrt{n}\|u\|_\infty, \quad u \in \mathbb{F}^n.$$

Every norm $\|\cdot\|$ on $\mathbb{F}^n$ is therefore also equivalent to the maximum norm with

$$\gamma_0 \max_{j=1,\ldots,n} |u_j| \leq \|u\| \leq \sqrt{n}\,\gamma_1 \max_{j=1,\ldots,n} |u_j| \tag{1.1.23}$$

for $u = (u_1, \dots, u_n) \in \mathbb{F}^n$. An arbitrary *sequence of vectors* $u^{(\iota)} = (u_1^{(\iota)}, \dots, u_n^{(\iota)})$, $\iota = 1, 2, \dots$ in the n-dimensional space $\mathbb{F}^n$ is said to *converge to a vector* u in $\mathbb{F}^n$ if the sequences of the components converge, i.e.,

$$u^{(\iota)} \to u \quad \Longleftrightarrow \quad u_j^{(\iota)} \to u_j, \quad j = 1, \dots, n, \quad \text{as} \quad \iota \to \infty. \tag{1.1.24}$$

By Theorem 1.1.1, the inequalities (1.1.23), and the relation (1.1.24), this happens if and only if, for an arbitrary norm on $\mathbb{F}^n$,

$$\|u^{(\iota)} - u\| \to 0 \quad \text{or equivalently} \quad \|u^{(\iota)} - u\|_\infty \to 0 \quad \text{as} \quad \iota \to \infty. \tag{1.1.25}$$

1.1.2 Matrix norms

The content of this subsection is taken almost verbatim from [74, Section 5.2]. We now consider $m \times n$ matrices $A = (a_{jk})$, $B = (b_{jk})$ with elements a_{jk}, b_{jk} in $\mathbb{F}$, $j = 1, \dots, m,\ k = 1, \dots, n$, where m and n are given natural numbers. These matrices again form a vector space over $\mathbb{F}$, addition of matrices and multiplication by a number λ in $\mathbb{F}$ being as usually defined by

$$A + B = (a_{jk} + b_{jk}), \quad \lambda A = (\lambda a_{jk}). \tag{1.1.26}$$

In this vector space of $m \times n$ matrices, we can introduce norms having the property $(N1) - (N4)$. These norms of matrices can be defined in various ways. For example, we can always regard $m \times n$ matrices (a_{jk}) as vectors (a_l) of the space $\mathbb{F}^N$ where $N = mn$, and conversely, every vector (a_l) in $\mathbb{F}^N$ can be associated with a matrix (a_{jk}), for instance, by the rule

$$a_l = a_{jk}, \quad l = j + m(k-1),\ j = 1, \dots, m,\ k = 1, \dots, n. \tag{1.1.27}$$

Remark. In this way, the column-wise storage of matrices is done in computer programs such as Matlab. □

In an N-dimensional space $\mathbb{F}^N$, every norm is equivalent to the maximum norm. We formulate this result for the norms of matrices in the following theorem.

Theorem 1.1.2. *(Equivalence of matrix norms in finite-dimensional spaces) Let m, n be a pair of natural numbers and $\|\cdot\|$ an arbitrary norm on the vector space $\mathbb{F}^{m\times n}$ of $m \times n$ matrices. Then, there exist positive numbers γ_0, γ_1 such that*

$$\gamma_0 \max_{\substack{j=1,\dots,m\\k=1,\dots,n}} |a_{jk}| \le \|A\| \le \gamma_1 \max_{\substack{j=1,\dots,m\\k=1,\dots,n}} |a_{jk}| \tag{1.1.28}$$

for every $m \times n$ matrix $A = (a_{jk}),\ a_{jk} \in \mathbb{F},\ j = 1, \dots, m,\ k = 1, \dots, n.$