Abstract

In this thesis we consider problems from nonlinear acoustics and fluid-structure interaction in a time-periodic framework.

We begin by studying two models from nonlinear acoustics, namely the Kuznetsov equation and the Blackstock-Crighton equation. Existence of time-periodic solutions to these systems are established for time-periodic data sufficiently restricted in size. We conclude that the dissipative effects in the Blackstock-Crighton equation and the Kuznetsov equation are sufficient to avoid resonance. The Blackstock-Crighton model is considered in a bounded domain with both non-homogeneous Dirichlet and Neumann boundary values, whereas the Kuznetsov equation is further studied in the whole space and in the half space. Existence of a solution is obtained via a fixed-point argument based on appropriate *a priori* estimates for the linearized equations. In order to deduce the L^q estimates, we decompose these systems into a stationary problem and a purely oscillatory problem, and consider the different Fourier modes separately. Via Fourier multiplier theory we obtain a strong time-periodic solution in an L^q framework. The investigation of these systems is carried out in Chapter 3.

In Chapter 4 the interaction of a viscous fluid with an elastic structure is studied. We consider a periodic cell structure filled with a viscous fluid, which interacts with the lower deformable boundary of the cell. The motion of the fluid is governed by the Navier-Stokes equitations and the deformable lower boundary is governed by the plate equation. Existence of a time-periodic solution to the linearized coupled system is deduced. Suitable L^q estimates are established for the linearized problem via Fourier multiplier theory and a localization argument. Finally existence of a solution to the nonlinear problem follow via a fixed-point argument.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit Problemen aus dem Bereich der nichtlinearen Akustik sowie Fluid-Struktur-Kopplungs-Problemen.

Aus dem Forschungsgebiet der nichtlinearen Akustik beschäftigen wir uns mit der Blackstock-Crighton-Gleichung und der Kuznetsov-Gleichung. Die Existenz einer zeitperiodischen Lösung dieser Modelle unter einfluß zeitperiodischer äußerer Kräfte wird gezeigt. Die Blackstock-Crighton-Gleichung wird in einem beschränkten Gebiet untersucht, wohingegen die Kuznetsov-Gleichung im Ganzraum, im Halbraum, sowie in einem beschränkten Gebiet betrachtet wird. Beide Modelle werden sowohl mit inhomogenen Dirichlet- als auch Neumann-Randbedingungen untersucht. Existenz von Lösungen für die beiden Systeme wird über ein Fixpunktargument hergeleitet. Hierzu benötigen wir L^q-Abschätzungen für die Lösung des zugehörigen linearen Systems. Die Abschätzungen werden mit Hilfe von Fourier-Multiplikatoren bewiesen. Die Untersuchung dieser beiden Modelle findet in Kapitel 3 statt.

In Kapitel 4 dieser Arbeit untersuchen wir die Interaktion eines viskosen Fluids mit einer elastischen Struktur. Hierzu nehmen wir an, dass sich die Flüssigkeit in einer periodischen Zelle befindet, welche einen deformierbaren Boden hat. Die Strömung wird durch die Navier-Stokes-Gleichungen beschrieben, und der deformierbare Boden ist eine dünne elastische Platte. Es wird gezeigt, dass eine zeitperiodische Lösung existiert, welche die Interaktion beschreibt, wobei die elastische Platte durch äußere periodische Kräfte angeregt wird. Unter Verwendung von Fourier-Multiplikatoren sowie einem Lokalisierungsargument erhalten wir L^q-Abschätzungen, welche wiederum benutzt werden, um ein Fixpunktargument durchzuführen. Mit Hilfe des Fixpunktarguments zeigen wir, dass eine zeitperiodische Lösung zum nichtlinearen freien Randwertproblem existiert.

1 Introduction

Resonance in fluid-structure interaction, as well as in the study of wave propagation, is a well-known phenomenon occurring in nature. Resonance can be observed when the frequency of an applied time-periodic force is in harmonic proportion to a natural frequency. The dynamic parameters such as displacement, velocity and energy of the system will then oscillate with increasing amplitude.

Subject of this thesis is to study the occurrence (or rather the absence) of resonance in different problems from nonlinear acoustics and fluid-structure interaction. Resonance occurs naturally in undamped hyperbolic systems, but damping mechanisms can prevent this. In the following, we study two types of damped system. First, we study the hyperbolic equations governing the propagation of an acoustic wave in a viscous medium, which introduces a damping effect. Second, we study a fluid-structure-interaction problem, where the hyperbolic equation governing the motion of the structure is damped via the interaction with a viscous fluid. In those cases, resonance can be avoided if the energy from the external forces accumulated over a period is dissipated via the damping mechanism. The existence of a time-periodic solution would be a manifestation hereof.

In nonlinear acoustics, the propagation of sound waves through a viscous medium is studied. An acoustic wave propagates through a medium as a local variation of pressure. Nonlinear effects occur when the waves exhibit high amplitudes. The *Blackstock-Crighton* equation and the *Kuznetsov equation* are typically used to model this type of nonlinear wave propagation. In [7], Blackstock first introduced the model

$$(a\Delta - \partial_t)\left(\partial_t^2 u - c^2\Delta u - b\partial_t\Delta u\right) - \partial_t^2\left(\frac{1}{c^2}\frac{B}{2A}(\partial_t u)^2 + |\nabla u|^2\right) = f, \tag{1.0.1}$$

which later was also derived by Crighton in [18]. This model is used to describe the motion of a wave when viscous, heat-conducting fluids are considered. However, if temperature constraints are neglected, the Blackstock-Crighton equation is reduced to a nonlinear damped wave equation

$$\partial_t^2 u - \Delta u - \frac{b}{c^2}\partial_t \Delta u - \partial_t\big(\frac{1}{\rho_0 c^4}\frac{B}{2A}(\partial_t u)^2 + |\nabla u|^2\big) = f, \tag{1.0.2}$$

called the Kuznetsov equation. This wave equation was first proposed by Kuznetsov in [57] and is a widely used model to describe the propagation of sound in fluids. In both the Blackstock-Crighton and Kuznetsov equation, the damping term is due to a Kelvin-Voigt damping $\partial_t \Delta u$, where u denotes the acoustic potential. The constant a is the heat conductivity of the fluid, c the speed of sound, and ρ_0 the mass density. The diffusivity of sound b is a measure of energy dissipation due to viscosity and heat conduction in the fluid. Finally, B/A denotes the so-called (acoustic) parameter of nonlinearity, which is the quotient of the second and first coefficient in the Taylor expansion of the pressure-density relationship, see [6]. Chapter 3 is devoted to the investigation of (1.0.1) and (1.0.2) under periodic forcing. More specific, given a force that is periodic in time with period $\mathcal{T}$,

$$f\colon \mathbb{R}\times\Omega \to \mathbb{R}, \qquad \forall (t,x)\in\mathbb{R}\times\Omega: \quad f(t+\mathcal{T},x) = f(t,x),$$

existence and L^q estimates of a $\mathcal{T}$-time-periodic solution u are studied. In case of the Kuznetsov equation, the whole space, half space and bounded domains are considered, whereas (1.0.1) is studied on a bounded domain. Inhomogeneous Boundary conditions of Dirichlet and Neumann type are examined.

The study of the interaction of a deformable structure with a viscous fluid is fundamental to many applications, for instance in the field of aeroelasticity, biomechanics or hydroelasticity, see for example [54, 78, 38]. In this thesis we carry out such a study for fluid-structure systems that are driven by a time-periodic force. Observe that the fluid domain, which is denoted by $\Omega_\eta(t)\subset\mathbb{R}^3$, changes in time, where η describes the evolution of the moving domain. The equations governing the motion of the fluid flow are given by the *Navier-Stokes* system

$$\begin{cases} \partial_t u - \mu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \operatorname{div} u = 0, \end{cases} \tag{1.0.3}$$

where u denotes the fluid velocity, p the associated pressure field and $\mu > 0$ a constant. As a model for the deformable structure, we consider a thin elastic plate, whose motion is governed by the *plate equation*

$$\partial_t^2\eta + \Delta'^2\eta - \nu\Delta'\partial_t\eta = F - \mathrm{T}_\eta, \tag{1.0.4}$$

where η is the displacement of the fluid-solid interface in transversal direction. The term T_η is the normal fluid stress tensor induced by the fluid on the elastic structure. Here, $\nu > 0$ is a constant. In Chapter 4, existence, regularity and uniqueness of a solution to (1.0.3)–(1.0.4) are established for suitable boundary values.

1.1 A Historical Background of Nonlinear Acoustics

Acoustics is the science of sound and is derived from the Greek word *akouein*, to hear. The study of sound goes back to the ancient world. Even then it was well known that sound propagates as a wave. However, the first experimental evidence of this phenomenon was in the seventeenth century. Most of the early acoustical investigations were closely tied to musical acoustics. It started with studying a vibrating string, over to a vibrating membrane, to the more complicated case of a vibrating plate. One of the first (theoretical) results in this field is due to D. Bernoulli, Euler and d'Alembert in the mid eighteenth century. Bernoulli introduced a partial differential equation for the vibrating string and gave a solution thereto which was interpreted by d'Alembert as a wave traveling in both directions along the string, see for example [68] and the references [5] and [21]. The one-dimensional model of the wave equation derived by d'Alembert played a fundamental role in fluid mechanics and elasticity, see [73, Section 2.1 and Section 2.2] for a more detailed description of the connection between fluid mechanics and acoustics. After the works of Bernoulli and d'Alembert, Euler has derived an equation for nonlinear plane acoustic waves in air, which described the behaviour of gas at constant temperature, see [29]. However, the correct law of describing the propagation of plane progressive waves was found hundred years later by Earnshaw. Mathematically, the works [63] and [74] of Lagrange and Poisson had an immense influence on solving nonlinear plane wave equations. Moreover, in the study of shock formation, Stokes already realized in 1848 that it is crucial to include viscosity in the description thereof. The main contribution here came from Rankine and Hugoniot, who first formulated the conservation laws (conservation equations for mass, momentum and energy) describing the connection of a flow field behind a shock with the flow field ahead of it. However, the first successful attempts to formulate a comprehensive model were made by Rayleigh and Taylor. All this together with the contributions of Fay (see for example [32]), and many other scientists in this field, leads to Burgers' equation (see [12]), which is still a classical model used to describe the propagation of plane waves. For more details on the historical evolution of acoustics before the middle of the twentieth century we refer to [68, 80].

After Lighthill published his article [67] in 1956, the interest in the study of acoustic waves grows. In [67] Lighthill describes propagation of shock waves, flood waves in rivers and traffic flow on highways, see [23, Chapter 1]. However, since there are nonlinear acoustical phenomena that can not be described adequately with Burgers' equation, more general wave equations are required. In [7] Blackstock introduced a special model of nonlinear wave equations of higher order, which models the propagation of sound in thermoviscous fluids. To be more precise, Blackstock followed the approach of Lighthill, which is based on the assumption that effects of nonlinearity and dissipation are small. Keeping linear and quadratic nonlinear terms that do not involve viscosity or heat conduction terms, Blackstock derived (1.0.1) from the full equations of motion for a thermoviscous flow, but without the damping term $\partial_t \Delta u$. This system further appears in Crighton's work [18]. If temperature constraints are neglected, the model leads to the Kuznetsov equation (1.0.2), which was first proposed by Kuznetsov in [57]. Moreover, Kuznetsov's equation is a generalization of d'Alembert's wave equation with new terms due to nonlinearity and dissipation.

The Blackstock-Crighton and Kuznetsov equtions have been subject to increasing research over the last years, see for example [52, 53, 70] where just recently well-posedness of the corresponding initial-value problem for the Kuznetsov equation (1.0.2) was established. Optimal regularity results for (1.0.1) and (1.0.2) were given in [11]. Moreover, the corresponding initial-value problem (1.0.1) was subject in [10, 9]. For more details on the mathematical investigation of the Kuznetsov and Blackstock-Crighton equation, we refer to [51], where a brief overview is given, and the references therein.

1.2 A Historical Background of Fluid-Structure Interaction

Broadly speaking, fluid-structure interaction denotes the coupling between the laws that describe the dynamics of a fluid and structural mechanics.

More specifically, it is the interactions between a deformable structure and a surrounding or internal fluid flow. At the fluid-structure interface, stress is exerted on the solid object by the fluid and leads to deformations hereof. Fluid-structure interaction is a widespread phenomenon in nature, for example in blood flow in human arteries, see [54, 78].

The modern investigation of fluid dynamics started in the middle nineteenth century when NAVIER first proposed a system of equations describing the motion of an incompressible viscous Newtonian flow. Independent of NAVIER's work, STOKES published (for the first time in a scientific article) in [85] the same model, which is nowadays still the most widely used system to model the motion of a liquid. The Navier-Stokes equations are given by

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } (0,T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega, \end{cases} \tag{1.2.1}$$

where the unknown $u\colon (0,T) \times \Omega \to \mathbb{R}^3$ and $p\colon (0,T) \times \Omega \to \mathbb{R}$ are the Eulerian velocity field and the pressure field of the liquid, respectively, and $f\colon (0,T) \times \Omega \to \mathbb{R}^3$ the external force. During this studies, STOKES further observed that the effective mass of a rigid body moving in a fluid increases. This phenomenon was first observed by BESSEL in 1828, when considering the motion of a pendulum in a fluid, and meant that the surrounding fluid increased the effective mass of the system. The observation of Stokes and his scientific contribution are known as the founding of (fluid mechanics and) fluid-structure interaction.

The first breakthrough in the mathematical analysis of the Navier-Stokes equations is due to LERAY in 1930. He showed existence of a weak solution to the Navier-Stokes equations, see [65, 66]. Later, HOPF [49] further developed this concept. In the context of time-periodic forcing, SERRIN [82] originally suggested to study time-periodic solutions to (1.2.1). However, the first complete results on existence of time-periodic solutions are due to PRODI [75], YUDOVICH [90] and PROUSE [76], who showed existence of weak time-periodic solutions. Over the past years, an increasing number of authors have investigated (1.2.1) in a time-periodic framework, see for example [35, 40, 41, 55, 60, 37, 59, 72]. When the domain $\Omega = \Omega(t)$ which occupies the fluid varies with time t, one of the first investigations is due to SATHER [81] and the work of FUJITA and SAUER [34].

In recent years the number of investigations into fluid-structure interaction have increased, see for example [39, 26, 25, 16, 19]. There are different types of fluid-structure-interaction problems. For example, one can consider a deformable body moving in a viscous fluid. These kind of free boundary problems were studied in [25]. Another case of fluid-structure interaction is to consider the fluid in a domain where the boundary or one part of it is an elastic structure like an elastic plate. This was subject to [16, 19]. In [16] existence of weak solutions to the initial-value problem corresponding to the coupled system (1.2.1) and

$$\partial_t^2 \eta + \Delta'^2 \eta - \nu \Delta' \partial_t \eta = f - \mathrm{T}_\eta \qquad \text{in } (0,T) \times \Gamma, \tag{1.2.2}$$

in a three-dimensional cavity is obtained. Here, $\eta\colon (0,T) \times \Gamma \to \mathbb{R}$ is the transversal displacement of the fluid-solid interface $\Gamma \subset \partial\Omega$, $\nu > 0$ a constant and T_η the normal component of the stress induced by the fluid on the elastic plate. A solution and some further L^2 estimates were established by presenting a weak formulation to the nonlinear problem and utilizing Galerkin approximation. da Veiga considered a similar model of the elastic plate interacting with a viscous fluid and showed existence of a strong solution in an L^2-framework by a fixed-point procedure. Recently, Denk and Saal [24] studied a similar model in the half space $\mathbb{R}^n_+$ where the boundary is given by a damped Kirchhoff plate model. The authors showed existence and uniqueness of a strong solution in an L^q-setting.

1.3 Nonlinear Acoustics with Periodic Forcing

To date, only the initial-value problem of the Blackstock-Crighton equation and Kuznetsov equation have been studied. In the following we carry out an investigation of the time-periodic version of these two problems. We start by considering the whole space case of the linearization of the Kuznetsov equation (1.0.2) given by

$$\partial_t^2 u - \Delta u + \partial_t \Delta u = f, \tag{1.3.1}$$

and establish *a priori* L^q estimates. Instead of relying on a Poincaré map, we obtain the estimates directly via a representation formula for the solution. We hereby circumvent completely the theory for the corresponding initial-value problem, and develop a more direct approach. Moreover, the representation formula we establish seems interesting in the context of resonance, since it exposes the way how different modes of the solution are damped in relation to the modes of the forcing term. To this end, we replace the time axis with the torus group $\mathbb{T} = \mathbb{R}/\mathcal{T}\mathbb{Z}$ and reformulate (1.3.1) as a partial differential equation on $\mathbb{T}$. In this setting, it is possible to utilize the Fourier transform $\mathscr{F}_G$ in a framework of tempered distributions $\mathscr{S}'(G)$, which yields the representation formula

$$u = \mathscr{F}_G^{-1}\left[\frac{1}{|\xi|^2 - k^2 + ik|\xi|^2}\mathscr{F}_G[f]\right] \tag{1.3.2}$$

for the solution. Here, $G := \mathbb{T} \times \mathbb{R}^n$, with $n \geq 2$. In order to obtain the desired L^q estimates, we decompose this formula into an "undamped" (the *steady state* part u_{s}) and "damped" (the *purely oscillatory* part u_{tp}) modes, *i.e.*,

$$u_{\mathrm{s}} + u_{\mathrm{tp}} = \mathscr{F}_G^{-1}\left[\frac{1}{|\xi|^2}\mathscr{F}_G[f]\right] + \mathscr{F}_G^{-1}\left[\frac{(1-\delta_{\frac{2\pi}{T}\mathbb{Z}}(k))}{|\xi|^2 - k^2 + ik|\xi|^2}\mathscr{F}_G[f]\right].$$

Since one mode is damped and the other is not, we cannot expect u_{tp} and u_{s} to have the same regularity. Actually, by decomposing the solution into u_{tp} and u_{s}, we see that the Fourier multiplier in the representation of u_{tp} leads to better L^q estimates than can be expected for u. Based on the results deduced for the whole space problem, we investigate (1.3.1) in the half space and on a bounded domain. In the half space case, the linearized damped wave equation (1.3.1) is studied by a reflection principle. In the case of a bounded domain, a localization argument yields the desired L^q estimates. Finally, existence of a time-periodic solution to the nonlinear Kuznetsov equation is established by a fixed-point argument.

The linearization of the Blackstock-Crighton equation (1.0.1) is given by

$$(a\Delta - \partial_t)\left(\partial_t^2 v - c^2\Delta v - b\partial_t\Delta v\right) = f, \tag{1.3.3}$$

and is considered in a bounded domain with both Dirichlet and Neumann boundary conditions. As for the Kuznetsov equation, we will decompose the linearized Blackstock-Crighton equation into a steady-state part and a purely oscillatory part. Instead of using Fourier multiplier theory to establish the desired *a priori* L^q estimates, we decompose (1.3.3) into a coupled system consisting of two equations of lower order, namely the time-periodic heat equation and the time-periodic wave equation. Based on known results for the heat and wave equations, we deduce existence of time-periodic solutions to (1.3.3). Similarly to the Kuznetsov equation, we obtain existence of a time-periodic solution to the nonlinear problem by utilizing a fixed-point argument.

1.4 Fluid-Structure Interaction with Periodic Forcing

Chapter 4 is devoted to the investigation of the Navier-Stokes equations (1.2.1) interacting with a thin elastic plate located at one part of the boundary, when a time-periodic external forcing is considered. The elastic structure is governed by the plate equation (1.2.2) in $\mathbb{T} \times \mathbb{T}_0^2$. Here, $\mathbb{T}$ denotes the torus given in the previous subsection and $\mathbb{T}_0^2 = (\mathbb{R}/L\mathbb{Z})^2$, with $L > 0$. The fluid problem and the solid problem are coupled by means of dynamic and kinematic interface conditions. For more details on the coupling we refer to Section 4.2. The coupled system (1.2.1) and (1.2.2) shall be studied in a layer domain $\Omega_\eta(t) = \mathbb{T}_0^2 \times (-\eta, 1)$, and the plate is located at the bottom of the domain. In the following we show existence of a solution to the coupled system. Instead of relying on a weak formulation of the free boundary problem, which yields solutions in an L^2 framework, we are interested in strong time-periodic solutions that obey an L^q estimate. To this end, we utilize a fixed-point argument, based on *a priori* estimates deduced for the corresponding linearized system. However, since the boundary of the fluid domain depends on the unknown η, a fixed-point argument cannot be utilized without any further modifications. For this reason, we first employ a coordinate transformation to reformulate the coupled system (1.2.1) and (1.2.2) on a *reference configuration*, where the boundary does not deform anymore. In this setting, we first consider the linearized system

$$\begin{cases} \partial_t^2\eta + \Delta'^2\eta - \nu\Delta'\partial_t\eta = F - e_3 \cdot \mathrm{T}(u,p)e_{3|x_3=0} & \text{in } \mathbb{T} \times \mathbb{T}_0^2, \\ \partial_t u - \mu\Delta u + \nabla p = f & \text{in } \mathbb{T} \times \Omega, \\ \operatorname{div} u = g & \text{in } \mathbb{T} \times \Omega, \\ u(t,x',0) = -\partial_t\eta(t,x')e_3 & \text{on } \mathbb{T} \times \mathbb{T}_0^2, \\ u(t,x',1) = 0 & \text{on } \mathbb{T} \times \mathbb{T}_0^2. \end{cases} \tag{1.4.1}$$

Existence of a solution to (1.4.1) is established via the concept of weak solutions to the corresponding resolvent problem. The *a priori* L^q estimates follow by a Fourier multiplier argument in combination with a localization argument similar to the case of nonlinear acoustics. Based on these results, a solution to the nonlinear problem is obtained via a fixed-point argument, and due to the transformation ϕ, a solution to the coupled system on the time-dependent domain.

2 Preliminaries

This chapter is dedicated to introduce the basic notation and concepts we will use during this doctoral thesis. It is convenient to formulate $\mathcal{T}$-time-periodic problems in a setting of function spaces where the torus $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ is used as a time axis. Indeed, via lifting with the quotient map $\pi\colon \mathbb{R} \to \mathbb{T}$, $\mathcal{T}$-time-periodic functions are canonically identified as functions defined on $\mathbb{T}$ and vice versa. Based on this observation we formulate the equations occurring in this thesis as systems defined on the torus $\mathbb{T}$ and decompose them into a so-called *steady-state* and *purely oscillatory* problem, which will be investigated separately. The necessity of this strategy will be obvious when investigating the regularity of a time-periodic solution. We will observe that the two parts of the solution solve various problems, and we will see that they do not have the same regularity properties. This concept of decomposing the time-periodic problems was first introduced in [58] and later generalized and further developed by Kyed and some co-authors, see for example [62].

Equipped with the quotient topology, the time-space domain $G := \mathbb{T} \times \mathbb{R}^n$ is a locally compact abelian group and therefore has a Fourier transform $\mathscr{F}_G$ associated to it, see Subsection 2.3. Moreover, in Subsection 2.2 we introduce the so-called *Schwartz-Bruhat space* $\mathscr{S}(G)$ and its dual space $\mathscr{S}'(G)$ of tempered distributions. In this framework we can formulate the partial differential equations on the group G and employ the Fourier transform $\mathscr{F}_G$, which will be introduced in the second part of this chapter, to obtain a solution. Moreover, we need two more tools from harmonic analysis to examine the resulting solution. We need a multiplier theorem to deduce the desired L^q estimates. But since classical multiplier theorems are defined in a whole space setting $\mathbb{R}^n$ we further have to introduce the Transference principle. The Transference principle is a useful tool that allows us to investigate the resulting multipliers in a whole space setting where we can make use of the known multiplier theory, and conclude the claims in the group setting from this. In 1965, de Leeuw was the first one introducing this principle (*cf.* [22]), which later was generalized by Edwards and Gaudry, see [27].

Furthermore, the function spaces on the torus $\mathbb{T}$ are introduced in Section 2.4. Since our approach is based on concept of Fourier multipliers we will propose all the Sobolev spaces via Bessel potential spaces, and show that these function spaces coincide with the general Sobolev spaces. However, the corresponding trace spaces, namely the Sobolev-Slobodeckiĭ spaces, are defined via real interpolation. Furthermore, we introduce the function spaces of solenoidal functions in Subsection 2.4.3.

In the framework of fluid-structure interaction Fourier multipliers on a boundary surface occur and we shall estimate the solution with the boundary data in the corresponding Sobolev-Slobodeckiĭ norm. In order to adapt the multiplier theory to the Sobolev-Slobodeckiĭ spaces, interpolation theory for function spaces of time-periodic functions has to be introduced. This will be done in Section 2.5. There, we will extend the known interpolation results to the concept of time-periodic functions.

In Section 2.6 the embedding and trace properties of time-periodic Sobolev spaces utilized in this doctoral thesis are introduced. These embedding properties are later employed to deduce the necessary L^q estimates to carry out a fixed-point argument, when investigating the nonlinear problems occurring herein.

Finally, in the last section of this chapter we introduce two useful mathematical tools from mathematical fluid mechanics. More precise, the Poincaré inequality on layer-like domains and the Bogovskiĭ operator are proposed. Note, Poincaré's inequality is not only used in the framework of fluid mechanics. But first we start by introducing some general notation utilized in this thesis.

2.1 General Notation

Throughout this thesis $\Omega \subset \mathbb{R}^n$ is always a domain with a sufficiently smooth boundary or the whole space $\mathbb{R}^n$. As long as not further mentioned, the space dimension $n \in \mathbb{N}$ is always greater than 2, where $\mathbb{N} := \{1, 2, \ldots\}$ denotes the set of all positive integers. If the set further contains zero, we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The domain Ω has a boundary of class $C^{k,1}$ or a $C^{k,1}$-smooth boundary, if $\partial\Omega$ can locally be expressed as the graph of a function $\omega \in C^k$ in the respective local coordinates, where the k-th order derivative of ω is Lipschitz continuous. That is, for any $x \in \partial\Omega$ there is a neighborhood $U \subset \mathbb{R}^n$ of x such that $\partial\Omega \cap \overline{U} \subset \mathrm{graph}(\omega)$. The outer unit normal vector on $\partial\Omega$ is denoted by ν, as far as not mentioned differently. Points in $\mathbb{T} \times \Omega$ are generally expressed by (t, x), with t being referred to as time, and x as the spatial variable. The time period $\mathcal{T} > 0$ remains